# Use and Misuse of Work-Energy Theorems[1]

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Some statements, such as

All generalizations are false

clearly have internal inconsistencies. Others may appear reasonable by themselves, but the inconsistencies become apparent when combined with related statements. Thus

$$W = \Delta(K.E.) \tag{1}$$

and

$$\Delta E_{\text{internal}} = Q + W \tag{2}$$

$$\text{(and hence } W = \Delta E_{\text{internal}} - Q \text{ )} \tag{2a}$$

often appear in the same textbook, without cautionary restrictions, although they are clearly incompatible with each other and with common physical examples. Each equation is valid for special conditions, but neither equation is general. (The same textbooks will also frequently define $W$ with one sign in mechanics and opposite sign in thermodynamics, without calling attention to the discrepancy.)

**Definition of Work**

We must start with a clear definition of work. Such a definition must satisfy (at least) two requirements: it must be consistent with the thermodynamic meaning of work implied by the first-law equation ($\Delta E = Q + W + ...$), and it must be compatible with the fuzzy, but important, concept of force times distance. This demands a two-part definition.

*a*. First, we require that work is a measure of energy transfer between a system and its surroundings. We choose the now more common sign definition, that $W$ is positive when energy moves from the surroundings to the system.[2]

*And*

*b*. To be consistent with part *a*, the "force times distance" portion of the definition requires more careful expression. Work is the sum, over all forces acting on the system,[3] of the integral of the scalar product of each force times the displacement of the point of application of that force.

$$W = \sum_i \left( \int \boldsymbol{f}_i \cdot \boldsymbol{dx}_i \right) \tag{3}$$

There are many examples in which work is done on a system although, because the forces are equal and opposite, the *net* force and therefore the displacement of the center of mass is zero. There are other examples in which the center of mass displacement is non-zero, but the point of application of the force does not move, so no work is done. And, of course, a center block in a moving (unaccelerated) string of blocks may do the same amount of work on the block in front as is being done on it by the block behind, so the *net* work is zero. The importance of integrating *each* force times the displacement of the point of application of that force was further brought forcibly to our attention by the paper of Sherwood and Bernard,[4] who showed that when friction forces act, the force acting on a macroscopic object times the displacement of the center of mass of that object does not measure work done (*i.e.*, energy transferred). This was elaborated upon by Bauman.[5]

Two-part definitions are hazardous. Nevertheless, there are no known exceptions to the double definition, and thus far no one has been able to show, for general problems, that the first part follows from the second. (We return to this point later.)

Work done on a system increases the energy of the system. It does *not* follow that if the energy of the system changes, then work is done on the system. The change in (total) energy of a system is equal to the sum given by the first-law equation,

$$\Delta E = Q + W + \textit{other terms} \tag{4}$$

so usually $\Delta E \neq W$, and clearly $\Delta E > 0$ does not imply $W > 0$ (or even $W \neq 0$).

**Limitations on Knowledge of *Q* and *W***

Work can only be measured, or calculated, in certain idealized processes. It is quite clear that work done on a system cannot be measured, or calculated, for a process involving friction. First, we don't know, in detail, the individual forces acting, or the displacements (or locations) of those forces. Second, the energy transferred does not (all) appear as mechanical energy in the system. Third, if we attempt to ignore the details and calculate work by applying the overall (*i.e.*, net) force and the overall (center of mass) displacement, we will find a violation of conservation of energy, because it is demonstrable that some energy appears as thermal energy,[6] in the system and in the surroundings, in addition to the energy put in and calculated as transferred as work.[7] The same arguments apply to energy transfers that involve internal friction and thus conversion of mechanical energy to thermal energy in both the system and the surroundings.

On the other hand, $\Delta E$ is, at least in principle, always subject to measurement and/or calculation. Because $\Delta E = Q + W$ in the absence of other modes of transfer, the inability to measure $W$ implies an inability to measure $\Delta E - W = Q$. Thus we may conclude that $Q$, also, can only be measured, or calculated, for certain idealized processes.

At the same time, we know that entropy is a state function, so $\Delta S$, like $\Delta E$, can always be known.  This provides the clue we needed.  Because $\Delta S = Q_{rev}/T$ (or, for an infinitesimal process, $dS = q_{rev}/T$), $Q_{rev}$ (or the infinitesimal value $q_{rev}$) can, in principle, always be known.  It follows that $W_{rev}$ (or $w_{rev}$) can also be known.

Thus we are led to the generalization that heat and work, $Q$ and $W$, can always be known for reversible, or equilibrium, processes, but cannot be assumed known for (thermodynamically) irreversible processes.  The obvious exceptions are those for which the quantities in question are zero.  For example, if a gas (ideal or nonideal) escapes through a pin hole into an (insulated) evacuated container, the forces exerted by the system on the surroundings are the forces exerted on the walls, which do not move (to a satisfactory approximation).  Therefore $W = 0$ and $Q = 0$, despite the irreversible character of the expansion.

In summary, we can measure, and calculate, $Q$ and $W$ for thermodynamically reversible processes, but for real processes (thermodynamically irreversible), we cannot know $Q$ or $W$, apart from special processes in which $Q$ and/or $W$ is necessarily zero.  For example, we know the floor does no work on the child who jumps ($W = 0$ and $Q = 0$).  We can find the work done by the child on a cord, as a block is pulled across the floor, and find the work done by the cord on the block, but cannot find the work done by the block on the floor (although we know the work is non-zero).

**Work and Kinetic Energy**

We have seen that an increase in kinetic energy[8] of the system provides no assurance of work being done on the system.  For example, as mentioned above, when you jump, you push down on the floor, without significantly moving the floor.  The floor does not supply any energy, so the floor does no work on you, although you change your kinetic energy.  You must supply your own energy to jump.[9]  An inelastic ("unhappy") ball dropped onto a table shows no detectable bounce. The kinetic energy (of the ball, associated with its center of mass) disappears, with no significant work done on or by the ball.

Older textbooks (pre-World War II) were typically satisfied with an equation such as

$$W = \int \boldsymbol{f} \cdot \boldsymbol{dx} \tag{5}$$

and a description of a force acting on a body through a distance.  (A cryptic comment in one text[10] that the force is the force applied by the agent doing the work and not, in general, the unbalanced force acting on the body, suggests that the authors already had second thoughts that prevented them from writing what has commonly been called the "work-energy theorem" and, in fact, they verbally deny its validity.)

**Particles and Other Objects**

More typical[11] of modern physics textbooks (post-World War II) is the statement

> "The work done by the resultant force acting on a particle is equal to the change in the kinetic energy of the particle."

That is,

$$W = \Delta K \tag{6}$$

This seems to be safe, because it is restricted to particles, but in the following chapter we find the principle restated as

> "To find the kinetic energy of a *body*, we compute the work done by the *resultant* force in setting the *body* in motion." [Emphases added.]

This suggests the lack of definition of "a particle" was not entirely accidental.

What *do* we mean by a particle? The answer is nontrivial.

Usually we interpret "particle" to mean a small body. In mathematics a *point particle* has zero size. Therefore mathematical particles cannot collide (*i.e.*, have infinitesimal chance of a collision). Thus gas particles cannot be treated as mathematical (point) particles, because we rely on collisions for equilibration. In nuclear physics a particle usually refers to one of the family of particles that come from nuclei and atoms: electrons, protons, neutrons, neutrinos, and the particles with very short lives, for example. Other times, a particle may be a bit of sand or a blood cell or some other bit of matter of what we might consider intermediate size.

In physics, particularly with respect to kinetic energy, it is convenient to introduce a distinctive definition of a particle, which we shall therefore call a "physical particle":

> A *physical particle* is an object of any size or shape that may have kinetic energy, but cannot (or does not in the processes we consider) change its energy in any way *except* a change of kinetic energy.

Hence planetary objects may be treated as particles to the approximation that tidal forces and collisions are ignorable. Atoms and molecules of a gas may be considered as physical particles to the approximation that there are no changes in rotational or vibrational states, or in low-lying electronic states.

Because *a physical particle may change only its kinetic energy*, physical particles may not change temperature, and thus physical particles cannot experience frictional forces, which have the effect of warming the particles affected. The work done on a physical particle (*i.e.*, the energy transferred to the particle) is equal to the change in kinetic energy of the particle, because the energy cannot go anywhere else.

Other particles may be small, *but cannot be considered to be physical particles*. They do not necessarily obey the work-kinetic energy theorem. Nor are they mathematical particles. They may, or may not, have negligible volumes. For example, the volumes of gas particles are measured by the van der Waals constant $b$, but this is not intrinsically related to whether there are low-lying states of the particles that may be excited or de-activated.

**A Valid General Relation for Work and Change of Energy**

If we restrict the discussion to what are here called *physical particles*, then the work-kinetic energy theorem as stated by equations 5 and 6,

$$W = \int \boldsymbol{f} \cdot \boldsymbol{dx} = \Delta(K.E.) \qquad (7)$$

is always valid, and the answer is the same whether we take the sum of the integrals of forces times their respective displacements or take the resultant force times the displacement of the particle. But for any body that does *not* satisfy the requirements of a physical particle, the results may be different, and the work-energy theorem, as customarily stated, is invalid.

An older textbook[12] states: "When ... work is done on a body, it can be entirely accounted for by one or more of the following effects: (1) *increase in the kinetic energy of the body,* (2) *increase in its potential energy,* or (3) *production of heat in opposing friction*." This is closer to the truth,[13] but does not allow for some other possibilities.

Allowing for the wide variety of means of energy storage in a body or collection of bodies, a good starting point is the "general work-energy theorem":

$$W = \Delta(K.E.) + \Delta(P.E.) + \Delta(\text{thermal energy}) + \Delta(\text{rotational energy}) + \Delta(\text{vibrational energy}) + \ldots$$

The general work-energy theorem may be considered as a composite of special cases — the "work-kinetic energy equation", the "work-potential energy equation", the "work-rotational-energy equation", the "work-thermal energy equation", and so forth.[14]

Keep in mind, however, that even the general work-energy equation is not fully general. It assumes:

*a*. Work is defined for the process considered; and

*b*. No energy transfer other than work.

(It also makes no specific allowance for pressure-volume contributions to energy. For some well-known processes, such as a Joule-Thomson expansion and Bernoulli's equation, we find that $W = \Delta H$.)

If we turn around the work-kinetic energy equation to get

$$\Delta(K.E.) = W \qquad (8)$$

it becomes more obvious that the generalization is invalid. Not only jumping off the floor, but change of kinetic energy caused by frictional forces, are well-known exceptions. In jumping, as we have seen, there is no (significant) displacement of the point of application of the force. In friction, the points of application of the forces are unknown and not simply related to the dynamics of the moving object.

**Work Contrasted with Pseudowork**

A bat is a pseudobird. It has some characteristics that make it look like a bird, but it is simply not a bird. We encounter a similar situation with equations that superficially look like an equation for work but do not give the actual work.

The issues raised here were extremely important in the historical development of physics. Newton developed physics solely from the point of view of momentum (and impulse, or $\Delta \boldsymbol{p}$). He was strongly challenged by Leibniz, who argued that *vis viva* ($mv^2$) was the important quantity.

Newton carefully avoided the trap that Leibniz tried to push him into. In particular, Newton avoided considerations of energy, and therefore of work. Yet much of the misunderstanding concerning work arises from similarity of the valid equation for work with the Newtonian equation that looks superficially like the equation for work. To make things more difficult, it is *sometimes* equal to work, and is therefore often confused with the equation for work. (At least a bat is *never* a bird.)

Following Newton's arguments involving forces and momenta, if a *net force* acts on a body, the *center of mass of the body* undergoes a (vector) displacement $\boldsymbol{dx}_{\mathrm{cm}}$.

$$\int \boldsymbol{f}_{\mathrm{net}} \cdot \boldsymbol{dx}_{\mathrm{cm}} = \int m\, \boldsymbol{a}_{\mathrm{cm}} \cdot \boldsymbol{dx}_{\mathrm{cm}} = \int m \frac{\boldsymbol{dv}_{\mathrm{cm}}}{dt} \cdot \boldsymbol{dx}_{\mathrm{cm}} \tag{9}$$

Here we can treat the derivative as a ratio of differentials. Assuming the displacement and velocity are in the same direction (usually satisfied for such problems), we can write a general condition for change in kinetic energy,

$$\int \boldsymbol{f}_{\mathrm{net}} \cdot \boldsymbol{dx}_{\mathrm{cm}} = \int m \frac{dv_{cm}}{dt} = \int m \frac{dx_{cm}}{dt}\, dv_{\mathrm{cm}} = \int m\, v_{\mathrm{cm}}\, dv_{\mathrm{cm}} = \frac{1}{2}\, m\, v^2 \tag{10}$$

Is this equation equivalent to the "work-kinetic-energy theorem" discussed above? No, usually not. The integral on the left-hand side is not necessarily equal to work done on the system and the right-hand side, $\Delta K$, may be only part of the change in energy. To find work, we must consider each *individual* force and the displacement of the point of application of each force. In general, that displacement will not be the same for all forces acting, and the displacement will not be the same as the displacement of the center of mass. We find the work done by each force, integrated over the displacement of the point of application of that force, then sum the work terms. In the momentum/kinetic energy equation,[15] we sum all the forces to find the *net force*, and integrate over the displacement of the *center of mass*. So this momentum-kinetic energy integral, despite the superficial resemblance to the integral for work, is usually *not* equal to work done on the system. The critical point is:

$$W = \sum_i \left( \int f_i \cdot dx_i \right) = ?\int \sum_{\mathrm{i}} \boldsymbol{f}_{\mathrm{i}} \cdot \boldsymbol{dx}_{\mathrm{cm}} = \Delta \left( \frac{1}{2} \mathrm{m}\, v^2 \right) \tag{11}$$

The left-hand side is equal to work; the right-hand side is equal to change in kinetic energy, but the left-hand side is not necessarily equal to the right-hand side.

For example, the force may act along a line that does not pass through the center of mass of the body. Then the body acquires rotational energy, in addition to the translational kinetic energy calculated here. The work done on the body includes energy appearing as translational energy and energy appearing as rotational energy. Or work may be done to change potential energy, with or without a change in kinetic energy. Also, the net force acting on an ideal gas compressed by a piston, acts through a distance that is not equal to the displacement of the center of mass of the gas, so work is done on the gas but there is no change in the kinetic energy of the gas (as a whole, and perhaps not even any change in the kinetic energy of the individual molecules, depending on what else is happening).

This momentum-kinetic energy equation does become equal to the work-kinetic energy equation *if* the system is a physical particle *or* if the forces acting are *body forces*, such as a uniform gravitational field, acting on all parts of the system equally. If the force is the same, and the displacement of the point of application of the force is the same, for all parts of the body, then net force times center of mass displacement is equal to the sum of forces times displacements. For other systems, the two equations usually give quite different values.

The *apparent* similarity of this momentum-kinetic energy equation to the expression for work has caused serious difficulties in the past. One must constantly be alert to the distinction to avoid erroneous conclusions.

**Energy and Momentum in Perspective**

From developments in the 20th century, we now recognize that energy is, indeed, the more important quantity. From energy we obtain the property known as mass (equal to total energy divided by $c^2$) and from energy we obtain the curvature of space-time we call force.[16] But we also can recognize, now, that the physics of energy is far more difficult. Most attempts to follow energy require substantial approximations.

Momentum is a conserved quantity, so as the momentum of a system changes it is relatively easy to discern where the momentum has gone; it always appears as momentum, mass times velocity.[17] Energy is also conserved, but energy changes form. There is no conservation law for mechanical energy. Mechanical energy is sometimes a constant of the motion, but that has nothing to do with conservation laws, which apply only to system + surroundings. When the energy of a system changes, that energy may appear in any of a variety of forms, or may simply be hiding from us within the system. Leibniz was doomed to failure by attempting to develop quantitative physics on the basis of *vis viva*, while Newton laid a firm foundation with equations for momentum, that permitted addition of the energy concept over the following centuries.

The importance is emphasized when energy is discussed in conjunction with ways in which energy is transferred between a system and its surroundings. Thermal energy transfer and work are two of the means for transferring energy, but each is defined only for processes in which they are equal to zero ($Q = 0$ or $W = 0$) *or* for processes that are fully reversible. Because real processes cannot be fully reversible, energy transfer processes are usually inherently approximations to reality.

In summary:  Be cautious about what you tell students about work and its consequences. Most generalizations are likely to apply only to special cases.  Or in other words, especially when discussing work:

"Most generalizations are false."

Endnotes:

1. Presented as Paper DE6 at *AAPT* Summer Meeting, Boise, Idaho, 2002.

2.  It has been observed that some physicists cringe at the introduction of the terminology of "system" and "surroundings" in a first course.  We could, of course, retain such terminology as "object" with the surroundings implied.  That becomes awkward when the "object" is actually many objects — a collection of molecules, or a liquid plus its container, or several interacting bodies.  The description as a *system* easily adapts to such common complex "objects".  The "surroundings" is then everything else in the universe, except the system, or to avoid irrelevant questions of cosmology, one may define the surroundings as everything in the universe, except the system, that is affected by the process under consideration.

3.  Forces acting on the system are necessarily what may be called "external" forces.  We do not include forces exerted by the system on itself, or "internal" forces, which cannot transfer energy from the surroundings to the system.

4. Bruce A. Sherwood and W.H. Bernard, *Am. J. Phys* **52**, 1001 (1984) and references given there.

5. Robert P. Bauman,  "Physics that Textbook Writers Usually Get Wrong", *Phys. Teach*. "I. Work",  **30**, 264-269, (May, 1992), and Robert P. Bauman, *Modern Thermodynamics with Statistical Mechanics*, Macmillan Publishing Co., New York, 1992.

6. Thermal energy is that (small) part of the internal energy that changes with change of temperature (or in a change of phase).

7.  R.P. Bauman, ref. 5.

8.  Unless otherwise specified, by kinetic energy we refer to kinetic energy of the system as a whole.  Obviously, internal kinetic energy may be increased by heating the system, with $W = 0$, or by letting parts act on each other, as when a compressed spring acting on two carts (all part of the system) is released, increasing the kinetic energy of the carts, with $W = 0$ for the system.

9.  If you choose to be picky and point out that the floor *does* deflect, and therefore does some work on you as you jump, you must still concede that any such $W$ is far less than the change in your kinetic energy.  Energy depends on the frame of reference, so the energies of you and the floor will vary with the frame of reference.  But for *any* frame of reference, *a*) there is no energy transfer from the floor to you (you can't skip breakfast by changing the frame of reference), and *b*) the force exerted by the floor does *not* act over the distance of your motion if you leave the floor.  The floor exerts a force on the soles of your feet only until the soles break contact with the floor, so the point of application of the force does not move when you jump.  (See also note 3.)

10. Erich Hausmann and Edgar P. Slack, Physics, 2nd edition, D. Van Nostrand, 1939; p. 128.

11. It is not the intention here to criticize individual textbooks because most follow the same common patterns.  Typical post-war texts must, of course, include Halliday and Resnick, but also many others.

12.  Hausmann and Slack, op. cit.; p. 131.

13.  For example, work done on an object may increase the rotational energy of the object.  Potential energy that is internal energy (as in a spring) causes no problems.  In contrast, although it is common practice to speak of "the potential energy of a particle", the label is misleading.  It is generally safer to assign the potential energy to the surroundings (the field in which the particle moves), although even this is an oversimplification.  The *potential* belongs to the field.  The *potential energy* is a product of the potential (of the field) and a property of the particle (typically mass or charge), and therefore belongs to *both* the field and the particle.  Nevertheless, because it is convenient to talk about work done on the particle (by the field), it is safer to follow the approximation of potential energy assigned to the field.

14. Just as kinetic energy refers to macroscopic kinetic energy, associated with motion of the center of mass, we let rotational energy here designate rotation of the body about the center of mass and vibrational energy designate macroscopic vibration about the center of mass.  Thermal energy is randomized internal energy, which changes with temperature.

Nearly every thermodynamics textbook defines "heat" as $Q$, the amount of thermal energy transferred to the system from the surroundings, but then consistently refers to thermal energy as "heat".  Although "heat" is a convenient label for many purposes, it is quite unsatisfactory as a technical term because there is no way for hearer or reader to know what is intended by the speaker or writer, or for the speaker or writer to know how the term will be interpreted by the hearer or reader.

15. The label "momentum/kinetic energy equation" is chosen to emphasize the Newtonian basis as (net) force acting on the center of mass and the consequent change in momentum and thus kinetic energy. In the two decades the topic has been under active examination, to the author's knowledge no one has come up with a better label for this equation. Certainly "pseudowork" is accurate only to the extent it is properly interpreted as "not work", but even then it is terribly non-specific; lots of things are not work. "Center-of-mass work" is worse, because the quantity is, in general, not work at all. Chabay and Sherwood retain the unfortunate *pseudowork* name but break the *system* into two superimposed models, a (physical) particle and the real system, to find the motion of the center of mass plus the detailed change in configuration of the real system and the various work terms applicable to the real system (Ruth W. Chabay and Bruce A. Sherwood, "Bringing atoms into first-year physics", *Am. J. Phys.* **67**, 1045-1050, December, 1999).

16. In general relativity, motions are tracked by accelerations determined by local curvature of space-time, so forces disappear. The curvature of space-time is determined by mass, equal to total energy divided by $c^2$.

17. As is well known, this is expressed for electromagnetic fields by $p = mv = mc = mc^2/c = E/c$.